Final exam: Math 232

GREEN SOLUTIONS

Instructions/Remarks:

- Read all instructions.
- The questions are on the 12 *numbered* single-sided pages following this one. Count them now.
- Please include all details and rough work. You may use the back of the question pages and there is extra paper should you require it.
- The exam is out of a total of 100 marks. The value of each question is indicated below.
- No calculators or electronic devices of any kind are permitted.
- You have 3 hours to complete this examination.
- Good luck!

Marks:

1) ________ /15 2) ________ /15 3) ________ /10 4) ________ /5 5) ________ /5

6) ________ /10 7) ________ /10 8) ________ /15 9) ________ /15

Total: ________ /100 Grade: ________

Name: ____________________ **SFU e-mail ID:** ____________________

Student Number: ____

Signature: ____

Question 1 [15pts] Suppose the entire population of a certain nation lives either in the city or the country. Consider the following model for how the population in the city and country changes year by year.

- Every year 10% of the people living in the city move to the country and the rest remain in the city.
- Every year 20% of the people living in the country move to the city and the rest remain in the country.

Let $x_1^{(k)}$ be the number of people living in the city in year k and let $x_2^{(k)}$ be the number of people living in the country in year k. Let $\mathbf{x}^{(k)} = \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix}$.

a. The model can be expressed as $\mathbf{x}^{(k+1)} = A\mathbf{x}^{(k)}$. What is A?

b. One of the eigenvalues of A is 1. What is the other eigenvalue?

c. Give an eigenvector corresponding to each eigenvalue of A.

d. Give an invertible matrix P and a diagonal matrix D, both of size 2×2 such that $AP = PD$.

Solution:

a. **[2 points]** $A = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}$.

b. **[5 points]** One way to get the answer is if they remember that the determinant of a matrix is the product of the eigenvalues. Since the determinant is $0.72 - 0.02 = 0.7$ we get that the other eigenvalue is 0.7. However, the more direct way is to compute the characteristic polynomial

$$p(\lambda) = \begin{vmatrix} 0.9-\lambda & 0.2 \\ 0.1 & 0.8-\lambda \end{vmatrix} = (0.9-\lambda)(0.8-\lambda) - 0.02 = 0.7 - 1.7\lambda + \lambda^2$$

and note (using the fact that $\lambda = 1$ is one root) that $p(\lambda) = (\lambda - 0.7)(\lambda - 1)$. So the other eigenvalue is 0.7.

c. **[5 points]** For $\lambda = 1$, find a nontrivial solution to $(A - I)\mathbf{x} = \mathbf{0}$.

$$A - I = \begin{bmatrix} -0.1 & 0.2 \\ 0.1 & -0.2 \end{bmatrix}.$$

$\mathbf{x} = (2, 1)$ works. For $\lambda = 0.7$, find a nontrivial solution to $(A - 0.7I)\mathbf{x} = \mathbf{0}$.

$$A - 0.7I = \begin{bmatrix} 0.2 & 0.2 \\ 0.1 & 0.1 \end{bmatrix}.$$

$\mathbf{x} = (1, -1)$ works.

d. **[3 points]** $P = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 \\ 0 & 0.7 \end{bmatrix}$ is one example.

Question 2 [15 pts] Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that reflects each point through the line $x = y$ and then stretches it in the x direction by a factor of 2.

a. What is the standard matrix for T?

b. What is the matrix for T relative to the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$?

c. State a basis $\mathcal{C}$ for which the matrix for T relative to the basis $\mathcal{C}$ is diagonal.

Solution:

a. **[4 points]**To construct the standard matrix we just need to compute $T(\mathbf{e}_1), T(\mathbf{e}_2)$. T takes $\mathbf{e}_1$, converts it to $\mathbf{e}_2$ and then the stretching doesn't change anything. T takes $\mathbf{e}_2$, converts it to $\mathbf{e}_1$ and then the stretching changes it to $2\mathbf{e}_1$. So

$$A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2)] = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$$

b. **[6 points]** First, given $\mathbf{x}$, note that

$$\mathbf{x} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} [\mathbf{x}]_{\mathcal{B}} = B[\mathbf{x}]_{\mathcal{B}}.$$

We need the inverse of B

$$[B|I] = \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & -1/2 & 1/2 \end{array}\right] = [I|B^{-1}].$$

$$\begin{aligned} [T]_{\mathcal{B}} &= [[T(\mathbf{b}_1)]_{\mathcal{B}} \quad [T(\mathbf{b}_2)]_{\mathcal{B}}] \\ &= B^{-1}AB \\ &= \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1/2 & 1 \\ 1/2 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3/2 & 1/2 \\ -1/2 & -3/2 \end{bmatrix} \end{aligned}$$

c. **[5 points]**Such a basis would consist of eigenvectors of A. First let's find the eigenvalues:

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 2 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 2.$$

So the eigenvalues are $\pm\sqrt{2}$. To get the eigenvector corresponding to $\sqrt{2}$ we need to find a vector in the null space of

$$A - \sqrt{2}I = \begin{bmatrix} -\sqrt{2} & 2 \\ 1 & -\sqrt{2} \end{bmatrix} \sim \begin{bmatrix} -1 & \sqrt{2} \\ 0 & 0 \end{bmatrix}.$$

So $(\sqrt{2}, 1)$ works. For $-\sqrt{2}$

$$A + \sqrt{2}I = \begin{bmatrix} \sqrt{2} & 2 \\ 1 & \sqrt{2} \end{bmatrix} \sim \begin{bmatrix} 1 & \sqrt{2} \\ 0 & 0 \end{bmatrix}.$$

So $(\sqrt{2}, -1)$ works. Therefore, let $\mathcal{C} = \left\{ \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}, \begin{bmatrix} \sqrt{2} \\ -1 \end{bmatrix} \right\}$.

Question 3 [10 pts] Consider the vectors

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}, \quad \mathbf{x}_4 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}.$$

a. Are these vectors linearly independent? Why or why not?

b. Find a subset of $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ that is a basis for $\text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$.

c. What is the dimension of $\text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$?

Solutions:

a. **[2 points]** No. Because $\mathcal{R}^3$ is a three-dimensional space, you cannot have a set of 4 linearly independent vectors in it. (Alernatively, they could discover this by row reducing the matrix, as in part (b) below.)

b. **[7 points]**Let

$$\begin{aligned} A = [\mathbf{x}_1 \ \ \mathbf{x}_2 \ \ \mathbf{x}_3 \ \ \mathbf{x}_4] \ &= \begin{bmatrix} 2 & 3 & 2 & 4 \\ 1 & -1 & 6 & 2 \\ 3 & 4 & 4 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 6 & 2 \\ 0 & 5 & -10 & 0 \\ 0 & 7 & -14 & 0 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & -1 & 6 & 2 \\ 0 & 1 & -2 & 0 \\ 0 & 7 & -14 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 6 & 2 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 & 2 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

So $\mathbf{x}_1, \mathbf{x}_2$ are a basis for $\text{Col}A = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$, since there is a pivot only in the first two columns of the resulting matrix.

c. **[1 points]** 2, since there are two vectors in the basis.

Question 4 [5 pts] Let $a, b, c, d, e, f, g, h, i$ be real numbers such that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 9.$$

State the following determinants. You do not need to justify your answer.

a.[**1 points**] $\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix} = 9.$

b.[**1 points**] $\begin{vmatrix} a & b & c \\ 3d & 3e & 3f \\ g & h & i \end{vmatrix} = 27.$

c.[**1 points**] $\begin{vmatrix} a & b & 0 \\ 3d & 3e & 0 \\ g & h & 0 \end{vmatrix} = 0.$

d.[**1 points**] $\begin{vmatrix} a-2d & b-2e & c-2f \\ d & e & f \\ g & h & i \end{vmatrix} = 9.$

e.[**1 points**] $\begin{vmatrix} 5a & 5b & 5c \\ d & e & f \\ g-a & h-b & i-c \end{vmatrix} = 45.$

Question 5 [5 pts] Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be a basis for $\mathbb{R}^3$. Let T be a linear transformation on $\mathbb{R}^3$ such that

$$T(\mathbf{b}_1) = \mathbf{b}_2 - \mathbf{b}_3, \quad T(\mathbf{b}_2) = \mathbf{b}_1 - \mathbf{b}_3, \quad T(\mathbf{b}_3) = \mathbf{b}_1 + \mathbf{b}_2.$$

a. What is $[T]_{\mathcal{B}}$, the matrix for T relative to the basis $\mathcal{B}$?

b. Is there a vector $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{0}$?

Solution:

a. **[3 points]**$[T]_{\mathcal{B}} = [[\mathbf{b}_1]_{\mathcal{B}} \;\; [\mathbf{b}_2]_{\mathcal{B}}] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}.$

b. **[2 points]**No. We can only have $T(\mathbf{x}) = \mathbf{0}$ if $[T]_{\mathcal{B}}$ is not invertible. But

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

so this is not possible.

Question 6 [10 pts] Find a basis for the null space of each of the following matrices:

a. **[4 points]** $\begin{bmatrix} 1 & 4 & 0 & -3 \\ 2 & 8 & 0 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

There are three free variables x_2, x_3, x_4. The general solution is $\mathbf{x} = \begin{bmatrix} -4x_2 + 3x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$. So a basis of the null space is $\left\{ \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

b.**[2 points]** $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. There is one free variable: x_1. So a basis for the null space is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

c.**[4 points]** $\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. There are two free variables: x_2, x_4. The general solution is $\mathbf{x} = \begin{bmatrix} 2x_2 \\ x_2 \\ 0 \\ x_4 \\ 0 \end{bmatrix}$. So a basis of the null space is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Question 7 [10 pts] In each case, state whether the matrix A is invertible or non-invertible and briefly justify your answer.

a. **[2 points]** $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 5 & -3 & 0 \end{bmatrix}$. Not invertible. Since two of the rows are identical, the rows are not linearly independent, so not invertible.

b. **[2 points]** $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Invertible since columns are linearly independent.

c. **[2 points]** A is a diagonalizable 4×4 matrix and has eigenvalues $1, 1, -i, i$. Invertible since no eigenvalues are 0.

d. **[2 points]** A is 3×3 and $A^T \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{0}$. Not invertible. Since the null space of A^T is non-trivial, A^T is not invertible and so A is not invertible.

e. **[2 points]**A is $n \times n$ and Col $A = \mathbb{R}^n$. Invertible by direct application of the Invertible Matrix Theorem.

Question 8 [15 pts] Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

a. Compute an orthonormal basis for Col A.

b. Let $\hat{\mathbf{b}}$ be the orthogonal projection of the vector $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ onto Col A. Compute $\hat{\mathbf{b}}$.

c. Find a least-squares solution of system $A\mathbf{x} = \mathbf{b}$.

Solution:

a. **[5 points]** Use the Gram-Schmidt procedure. Let $\mathbf{u}_1 = \mathbf{v}_1 = \mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Then

$$\mathbf{v}_2 = \mathbf{a}_2 - (\mathbf{a}_2 \cdot \mathbf{u}_1)\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

We normalize this to let $\mathbf{u}_2 = (0, 1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})^T$. Finally,

$$\mathbf{v}_3 = \mathbf{a}_3 - (\mathbf{a}_3 \cdot \mathbf{u}_1)\mathbf{u}_1 - (\mathbf{a}_3 \cdot \mathbf{u}_2)\mathbf{u}_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \sqrt{3}\begin{bmatrix} 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}.$$

So let $\mathbf{u}_3 = (0, 1/\sqrt{2}, 0, -1/\sqrt{2})^T$. So the answer is

$$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} \right\}$$

b. **[5 points]**

$$\begin{aligned} \hat{\mathbf{b}} &= (\mathbf{b} \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{b} \cdot \mathbf{u}_2)\mathbf{u}_2 + (\mathbf{b} \cdot \mathbf{u}_3)\mathbf{u}_3 \\ &= 0\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{3}}\begin{bmatrix} 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} - \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 1/3 - 1/2 \\ 1/3 \\ 1/3 + 1/2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1/6 \\ 1/3 \\ 5/6 \end{bmatrix} \end{aligned}$$

c. **[5 points]** We need to find a solution to the system given by the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 2 & -1/6 \\ 0 & 1 & 1 & 1/3 \\ 0 & 1 & 0 & 5/6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 2 & -1/6 \\ 0 & 0 & -1 & 1/2 \\ 0 & 0 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 2 & -1/6 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 5/6 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/6 \\ 0 & 1 & 0 & 5/6 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So $\mathbf{x} = \begin{bmatrix} 1/6 \\ 5/6 \\ -1/2 \end{bmatrix}$.

Question 9 [15 pts] Suppose A has eigenvalues $\lambda = 0.9$ and $\lambda = 1.1$ with corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

a. **[3 points]** Give diagonal D and invertible P such that $A = PDP^{-1}$.

b. **[7 points]** Find a formula for A^{1000}. (Your answer should be a 2×2 matrix with entries of the form $C_0 a^{1000} + C_1 b^{1000}$ where C_0, C_1, a, b are constants.)

c. **[5 points]** Let $x^{(k+1)} = Ax^{(k)}$ where $x^{(0)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. For large k there is an approximate formula for $x^{(k)}$:

$$x^{(k)} \approx \gamma^k \begin{bmatrix} c \\ d \end{bmatrix},$$

where γ, c, d are constants. State γ, c, d.

Solution:

a. $D = \begin{bmatrix} 0.9 & 0 \\ 0 & 1.1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

b. $A^{1000} = PD^{1000}P^{-1}$ We need to compute P^{-1}.

$$[P|I] = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & 1/2 & -1/2 \end{array}\right] = [I|P^{-1}].$$

So

$$\begin{aligned} A^{1000} &= PD^{1000}P^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0.9^{1000} & 0 \\ 0 & 1.1^{1000} \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \\ &= \begin{bmatrix} 0.9^{1000} & 1.1^{1000} \\ 0.9^{1000} & -(1.1^{1000}) \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}0.9^{1000} + \frac{1}{2}1.1^{1000} & \frac{1}{2}0.9^{1000} - \frac{1}{2}1.1^{1000} \\ \frac{1}{2}0.9^{1000} - \frac{1}{2}1.1^{1000} & \frac{1}{2}0.9^{1000} + \frac{1}{2}1.1^{1000} \end{bmatrix} \end{aligned}$$

c.

$$\begin{aligned} \mathbf{x}^{(k)} &= A^k \mathbf{x}^{(0)} = \begin{bmatrix} \frac{1}{2}0.9^k + \frac{1}{2}1.1^k & \frac{1}{2}0.9^k - \frac{1}{2}1.1^k \\ \frac{1}{2}0.9^k - \frac{1}{2}1.1^k & \frac{1}{2}0.9^k + \frac{1}{2}1.1^k \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{2}0.9^k + \frac{1}{2}1.1^k \\ \frac{3}{2}0.9^k - \frac{1}{2}1.1^k \end{bmatrix} \approx \begin{bmatrix} \frac{1}{2}1.1^k \\ -\frac{1}{2}1.1^k \end{bmatrix} \end{aligned}$$

for large k. So $\gamma = 1.1$, $c = 1/2$, $d = 1/2$. (They don't need to explicitly state γ, c, d if the get the expression correct.)